1AEDP – EXTRA QUESTIONS

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1. Banach Spaces

1.1. Let (f_n) be the sequence defined by (2.5). The lecture notes use (f_n) to show that $\big(C^0([0,1]), \|\cdot\|_{L^1}\big)$ is not a Banach space. Show, directly from the definition, that (f_n) is *not* Cauchy in $\big(C^0([0,1]), \|\cdot\|_{C^0([0,1])}\big)$.
(We can also argue that, since $\big(C^0([0,1]), \|\cdot\|_{C^0([0,1])}\big)$ is a Banach space, if (f_n) was Cauchy, then the limiting function f satisfying $\sup_{x\in[0,1]} |f_n(x) - f(x)| \to 0$ must be continuous, a contradiction).

1.2. The Fermi–Dirac function is defined by $F_\beta(x) = \big(1 + e^{\beta(x-\mu)}\big)^{-1}$ where $\beta \geqslant 0$ is the inverse Fermi-temperature and $\mu \in \mathbb{R}$ is the chemical potential. Is $(F_\beta)_{\beta\in\mathbb{N}}$ Cauchy in $\big(C^0([0,1]), \|\cdot\|_{C^0([0,1])}\big)$? What about $\big(C^0([0,1]), \|\cdot\|_{L^1}\big)$?

1.3. Is $(x \mapsto x^n)$ Cauchy in $\big(C^0([0,1]), \|\cdot\|_{C^0([0,1])}\big)$?

2. Change of Variables in Integrals

2.1. Use polar coordinates to evaluate

$$\int_0^\infty e^{-x^2} \mathrm{d}x.$$

Hence show that $f(x) := \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ is a probability distribution function on $\mathbb{R}$ (i.e. show that it is positive and integrates to 1). Write down a recurrence for the *moments* $\mu_n := \int x^n f(x)\mathrm{d}x$.

2.2. What is the area of an Ellipse

$$\left\{(x,y) \in \mathbb{R}^2 \colon \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leqslant 1\right\}.$$

2.3. Calculate the volume of the Ellipsoid

$$\left\{(x,y,z) \in \mathbb{R}^3 \colon \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 \leqslant 1\right\}.$$

3. Hilbert Spaces

3.1. Let $\mathbb{R}^{N\times N}_{\text{sym}} \subset \mathbb{R}^{N\times N}$ denote the space of symmetric $N \times N$ matrices. *(i)* Show that $\mathbb{R}^{N\times N}$ with inner product $(A,B)_{\mathrm{F}} := \mathrm{Tr} A^{\mathrm{T}} B$ (known as the *Frobenius inner product*) is a Hilbert space. *(ii)* Show that $\mathbb{R}^{N\times N}_{\text{sym}}$ is a closed linear subspace. *(iii)* What is the orthogonal projection $P_{\mathbb{R}^{N\times N}_{\text{sym}}}$?

3.2. Fix $u \in V$. Show that $\varphi_u \colon V \to \mathbb{K}$ defined by $v \mapsto (u,v)_V$ is a bounded linear functional on V with norm $\|\varphi_u\|_{\mathcal{L}(V,\mathbb{K})} = \|u\|_V$.
[This is the reverse of the Riesz representation theorem]

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4. Caratheodory construction of measures

A measure on an *algebra* $\mathcal{A}$ is a map $\mu_0 \colon \mathcal{A} \to \overline{\mathbb{R}}$ with $\mu_0(A) \geqslant 0$, $\mu_0(\emptyset) = 0$, and $\mu_0\left(\bigcup_n A_n\right) = \sum_n \mu_0(A_n)$ for a (countable) sequence of pairwise disjoint elements $A_n \in \mathcal{A}$.

[this is sometimes called a *pre-measure*]

Theorem 4.1 (Caratheodory Extension Theorem)**.** *Suppose $\mu_0 \colon \mathcal{A} \to \overline{\mathbb{R}}$ is a measure on the algebra $\mathcal{A}$. There exists a measure $\mu \colon \sigma(\mathcal{A}) \to \overline{\mathbb{R}}$ extending μ_0 to the σ-algebra generated by $\mathcal{A}$. If μ_0 is σ-finite, then μ is unique.*

4.1. Define $\mu^\star \colon \mathcal{P}(X) \to \overline{\mathbb{R}}$ by

$$\mu^\star(A) := \inf\left\{ \sum_n \mu_0(A_n) \colon (A_n)_{n=1}^\infty \subset \mathcal{A}, A \subseteq \bigcup_n A_n \right\}. \tag{4.1}$$

Show that *(i)* $\mu^\star(\emptyset) = 0$, *(ii)* $A \subset B \implies \mu^\star(A) \leqslant \mu^\star(B)$, and *(iii)* $(A_n)_{n=1}^\infty \subset X \implies \mu^\star\left(\bigcup_n A_n\right) \leqslant \sum_n \mu^\star(A_n)$.

[any map satisfying these properties is called an *outer measure*]

4.2. Define

$$\mathcal{M} := \{B \subseteq X \colon \mu^\star(A) = \mu^\star(A \cap B) + \mu^\star(A \setminus B) \quad \forall A \subseteq X\}. \tag{4.2}$$

Show that $\mathcal{M}$ is a σ-algebra.

[$B \in \mathcal{M}$ is known as $\mu^\star$-*measurable*]

4.3. Show that $\mu := \mu^\star\big|_{\mathcal{M}} \colon \mathcal{M} \to \overline{\mathbb{R}}$ is a measure.

4.4. Show that $\mathcal{A} \subseteq \mathcal{M}$ and thus $\sigma(\mathcal{A}) \subseteq \mathcal{M}$.

4.5. It is possible to show (but is more difficult) that if ν is another measure on $\mathcal{M}$ with $\nu(A) = \mu(A)$ for all $A \in \mathcal{A}$ then we actually have:

$$\nu(A) \leqslant \mu(A) \quad \text{for all } A \in \mathcal{M}, \tag{4.3}$$

$$A \in \mathcal{M} \text{ with } \mu(A) < \infty \implies \nu(A) = \mu(A). \tag{4.4}$$

Use this to show that if there exists a countable sequence $(A_n) \subset \mathcal{M}$ with $X = \bigcup_n A_n$ and $\mu(A_n) < \infty$ [i.e. if μ is σ-*finite*], then $\nu(A) = \nu(A)$ for all $A \in \mathcal{M}$.

4.6. Suppose $\mathcal{A}$ is the algebra of finite unions of open intervals on $\mathbb{R}$ and define $\mu_0(\emptyset) = 0$ and $\mu_0(A) = +\infty$ if $A \neq \emptyset$. Show that the extension of μ_0 to the Borel σ-algebra is not unique. Why does this not contradict the conclusions of the theorem?

5. Measure Theory

5.1. Assume that Fatou's lemma is satisfied (you can prove this directly without using the monotone convergence theorem). Prove the montone convergence theorem using Fatou's lemma.

5.2. Suppose $\varphi \geqslant 0$ is a smooth, compactly supported function with $\int_{\mathbb{R}^d} \varphi(x)\mathrm{d}x = 1$ (integral with respect to the Lebesgue measure in $\mathbb{R}^d$). Define $\varphi_n(x) := n^\alpha \varphi(nx)$. What choice of α gives $\int \varphi_n = 1$ for all n? Suppose that f is a continuous function. Show that $\int_{\mathbb{R}^d} f(x)\varphi_n(x)\mathrm{d}x \to f(0)$ as $n \to \infty$.

6. Quiz

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1. True or false: $C^0([a,b])$ with the norm $\int_a^b |f(x)|\mathrm{d}x$ is complete?

A. True	B. False

2. True or false: $\|f\| := \int |f'(x)|\mathrm{d}x$ defines a norm on $C^0([a,b])$?

A. True	B. False

3. Suppose $\|\cdot\|_1, \|\cdot\|_2$ are norms on a vector space V. Which of the following is not a norm?

A. $\|v\|_A := \|v\|_1 + \|v\|_2$	B. $\|v\|_D := \left(\|v\|_1^{1/2} + \|v\|_2^{1/2}\right)^2$
C. $\|v\|_B := \max_{j=1,2} \|v\|_j$	D. $\|v\|_D := (\|v\|_1^{2024} + \|v\|_2^{2024})^{\frac{1}{2024}}$

4. Which of the following inequalities holds for all $f \in C^0([0,1])$?

A. $\|f\|_{C^0([0,1])} \leqslant \|f\|_{L^1}$	B. $\|f\|_{C^0([0,1])} \geqslant \|f\|_{L^1}$
C. Neither A or B	

5. Suppose $T\colon V \to W$ is a linear operator between normed vector spaces. Which is the odd one out?

A. T is continuous at 0	B. $\|Tu\|_W < \infty$ for all $u \in V$
C. T is continuous	C. $\|T\|_{\mathcal{L}(V,W)} < \infty$

6. Take $V = \mathbb{R}^d$ which is a Hilbert space with $(u,v) := \sum_i u_i v_i$. Let $K = \{x \in \mathbb{R}^d \colon |x| = 1\}$. What is $P_K u$?

A. Not defined because K is not a closed linear subspace.	B. $P_K u = \frac{u}{\|u\|}$
C. $P_K u = u$ if $\|u\| \leqslant 1$ and $P_K u = \frac{u}{\|u\|}$ otherwise	D. Something else

7. Riesz representation theorem states

A. $\liminf \int f_n \geqslant \int \liminf f_n$	B. For every $u \in V$, there exists a unique $\phi \in V'$ such that $(u,v) = \phi(v)$ for all $v \in V$.
C. Every $\sigma-$finite measure defined on an alegbra $\mathcal{A}$ can be uniquely extended to a measure on $\sigma(\mathcal{A})$.	D. For every $\phi \in V'$, there exists unique $u \in V$ such that $\phi(v) = (u,v)$ for all $v \in V$.

8. True or False: For a measure μ and measurable sets A, B, we always have $\mu(A \cap (X \setminus B)) = \mu(A) - \mu(B)$?

A. True	B. False

9. Let $f(x) := x$ if $x \notin \mathbb{Q}$ and $f(x) = 0$ if $x \in \mathbb{Q}$. What is $\int_{[0,1]} f(x) \mathrm{d}x$ (integration with respect to Lebesgue measure)?

A. Undefined: f is not measurable	B. 0
C. 1	D. $\frac{1}{2}$

10. Which way round is Fatou's lemma? (f_n sequence of non-negative measurable functions)

A. $\liminf \int f_n \leqslant \int \liminf f_n$	A. $\liminf \int f_n \geqslant \int \liminf f_n$

11. Let $f(x) := x$ if $x \notin \mathbb{Q}$ and $f(x) = 0$ if $x \in \mathbb{Q}$. What is $\int_{[0,1]} f(x) \mathrm{d}x$ (integration with respect to Lebesgue measure)?

A. Undefined: f is not measurable	B. 0
C. 1	D. $\frac{1}{2}$

12. How are you feeling about the exam?

A. Terrible	B. Not great
C. It'll be okay	D. Very confident!

7. Fourier Series

For a function $u \in L^1((-\pi,\pi);\mathbb{C})$, we define $\tilde{u}$ to be its periodic extension to $\mathbb{R}$,

$$c_n(u) := \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} u(x)e^{-inx}\mathrm{d}x \quad \text{and} \quad S_N u := \sum_{n=-N}^{N} c_n(u)e^{inx}. \tag{7.1}$$

Essentially because $\left((2\pi)^{-\frac{1}{2}}e^{inx}\right)_{n\in\mathbb{Z}}$ forms a Hilbert basis, we have $S_N(u) \to u$ in $L^2(-\pi,\pi)$ (Theorem 6.2).

7.1. Theorem 6.1 states that, for $u, v \in L^1(-\pi,\pi)$, we have $u = v$ if and only if $c_n(u) = c_n(v)$ for all n. Show that if $u \in L^1(-\pi,\pi)$ with $\sum_{n\in\mathbb{Z}} |c_n(u)| < \infty$, then $\lim_{N\to\infty} S_N(x) = u(x)$ almost everywhere. In particular, in this case, u is almost everywhere equal to a continuous function,

7.2. Use the proof of the Riemann-Lebesgue lemma to show that, for $\tilde{u} \in C^{k,\alpha}$ with $k \geqslant 0$ and $0 < \alpha \leqslant 1$, we have $|c_n(u)| \leqslant Cn^{-(k+\alpha)}$. For which (k,α) does $S_N(u) \to u$ pointwise?

[It turns out that using a different proof, $S_N(u) \to u$ pointwise when $\tilde{u} \in C^{0,\alpha}$. The proof of this fact uses the Dirichlet kernel – see question 7.5]

7.3. An alternative proof of the Reimann–Lebesgue lemma: Suppose P is a trigonmetric polynomial (P is a finite linear combination of e^{inx} for $n \in \mathbb{Z}$). Show that $c_m(P) = 0$ for all sufficiently large $|m|$. Use the fact that trigonometric polynomials are dense in $L^1(-\pi,\pi)$ to conclude.

7.5. Show that $(u \star e_n)(x) = c_n(u)e^{inx}$ and hence write down an expression for the *Dirichlet kernel*, D_N, satisfying $S_N u = D_N \star u$. First person to read this and let me know will win some sweets next time we meet. Show that $\int_{-\pi}^{\pi} D_N(x)\mathrm{d}x = 1$ and simplify the expression for D_N to get $D_N(x) = A\frac{\sin(B_N x)}{\sin Cx}$ for some constants A, B_N, C]

7.6. Show that

$$\begin{aligned} S_N u(0) - u(0) &= \int_{-\pi}^{\pi} \big(u(-x) - u(0)\big) D_N(x)\mathrm{d}x \\ &= \int_{-\pi}^{\pi} \big(u(-x) - u(0)\big) \left[\frac{\sin(2N\pi x)\cos\pi x}{\sin\pi x} + \cos(2N\pi x)\right] \mathrm{d}x. \end{aligned} \tag{7.2}$$

Suppose that $u \in L^1$ and u is differentiable at 0. Show that $S_N u(0) \to u(0)$. Adapt this argument to show that $S_N u(0) \to \frac{1}{2}[u(0^-) + u(0^+)]$ when u is piecewise C^1.

7.7. Show that $u(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left[(2^{n^3}+1)\frac{x}{2}\right]$ is a continuous function with $S_N u(0) \to \infty$ as $n \to \infty$.

I have moved this question here because it is harder than I thought (and also uses results for the Dirichlet kernel)!

7.4. (more difficult) We now show that, for a given sequence of positive numbers a_n with $(a_n) \to 0$ as $n \to \pm\infty$ such that $a_n = a_{-n}$ and $a_n \leqslant \frac{1}{2}(a_{n-1} + a_{n+1})$ for $n > 0$, there exists a function $u \in L^1(-\pi,\pi)$ such that $c_n(u) = a_n$. That is, the Fourier coefficients of an integrable function can decay arbitrarily slowly,

7.4.1. First suppose that $a_n = a_{-n}$ and that $a_n \leqslant \frac{a_{n-1}+a_{n+1}}{2}$ for $n > 0$. Define $d_n := a_{n-1} - a_n$. Show that d_n is decreasing, $\sum_{n=1}^{\infty} d_n$ converges and calculate its limit, and show that $nd_n \to 0$ as $n \to \infty$.

7.4.2. Hence, show that $\sum_{n=1}^{\infty} n(a_{n-1} + a_{n+1} - 2a_n)$ converges and calculate its limit.

7.4.3. Show that the *Fejer* kernel $F_n := \frac{1}{n}\sum_{m=0}^{n-1} D_m$ is given by

$$F_n(x) = \frac{1}{n}\left(\frac{\sin\frac{nx}{2}}{\sin\frac{x}{2}}\right)^2 = \sum_{|m|\leqslant n-1}\left(1-\frac{|m|}{n}\right)e^{imx}$$

7.4.4. Using the fact that $\|F_N\|_{L^1(-\pi,\pi)} = 1$ (why is this true?), show that

$$u(x) := \sum_{n=1}^{\infty} n(a_{n-1}+a_{n+1}-2a_n)F_n(x)$$

converges absolutely in L^1.

7.4.5. Show that

$$c_m(u) = \sum_{n\geqslant|m|}^{\infty} n(a_{n-1}+a_{n+1}-2a_n)\left(1-\frac{|m|}{n}\right) = a_{|m|}$$

In summary, for $u \in L^2$, we have convergence of the Fourier series in L^2 and so, along a subsequence, we get pointwise convergence almost everywhere. In fact, one may extend this to get pointwise almost everywhere convergence, but the proof is difficult [Carleson]. For piecewise C^1 functions, we get almost everywhere convergence with Gibbs oscillations. Finally, $\tilde{u} \in C^0$ is **not** sufficient to get pointwise convergence, but $\tilde{u} \in C^1$ is. In fact, one can show that $\|D_N\|_{L^1} \sim C\log N$ which leads to: for $\tilde{u} \in C^{0,\alpha}$,

$$|u(x) - S_N u(x)| \leqslant C\frac{\log N}{N^\alpha} \to 0 \qquad \text{as } N \to \infty.$$

The smoother the (periodic extension of) function, the faster the Fourier coefficients decay. Fourier coefficients may decay arbitrarily slowly for general L^1 functions. When, u extends to an analytic function in the complex plane, the Fourier coefficients decay exponentially....

8. (Schwartz) Distributions

Recall $\Omega \subset \mathbb{R}^d$ is open. We say that $T \in \mathcal{D}'(\Omega)$ vanishes on $U \subset \Omega$ if $T(\varphi) = 0$ for all $\varphi \in \mathcal{D}(\Omega)$ with $\operatorname{supp}\varphi \subset U$. Let $\mathcal{U}$ be the union of all open sets on which T vanishes. Then, $\operatorname{supp} T = \Omega \setminus \mathcal{U}$. Suppose that $0 \in \Omega$ and consider $\delta_0 \in \mathcal{D}'(\Omega)$.

8.1. Show that $\operatorname{supp}\partial^\alpha\delta_0 = \{0\}$.

We now show a converse of this result and characterise the set of all distributions with support $\{0\}$.

8.2. Take $\eta \in \mathcal{D}(\Omega)$ with $\eta = 1$ in a neighbourhood of 0. Why is $T(\psi) = T(\eta\psi)$ for all $\psi \in \mathcal{D}(\Omega)$? Hence, show that T has finite order $p \geqslant 0$.

8.3. Use Taylor's theorem to write $\varphi \in D(\Omega)$ as $\varphi(x) = \sum_{|\alpha|\leqslant p}\frac{\partial^\alpha\varphi(0)}{\alpha!}x^\alpha + \theta(x)$ with $\partial^\alpha\theta(0) = 0$ for all $|\alpha| \leqslant p$. Hence, show that

$$T(\varphi) = \sum_{|\alpha|\leqslant p}\frac{\partial^\alpha\varphi(0)}{\alpha!}T(x^\alpha) + T(\eta\theta).$$

8.4. Explain why $|T(\eta\theta)| \leqslant C|\operatorname{supp}\eta|$,

8.5. In particular, since η is arbitrary (as long as $\eta = 1$ in a neighbourhood of 0), show that

$$T = \sum_{|\alpha|\leqslant p} c_\alpha\partial^\alpha\delta_0, \qquad \text{where} \qquad c_\alpha := \frac{(-1)^{|\alpha|}}{\alpha!}T(\eta x^\alpha). \tag{8.1}$$

8.6. In general, if $\operatorname{supp} T = \{x_n\}_{n=1}^N$, then T has finite order $p \geqslant 0$ and is a finite linear combination of $\partial^\alpha \delta_{x_n}$ for $|\alpha| \leqslant p$ and $1 \leqslant n \leqslant N$.

9. Quiz

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1. What is the order of $\delta_0 + \delta_0' \in D'(\mathbb{R})$?

A. not defined	B. 0
C. 1	D. 2

2. What is the derivative of $T_{|x|} \in D'(\mathbb{R})$?

A. not defined	B. 1
C. $T_{\text{sgn}}, \quad \text{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$	D. something else

3. Suppose $T \in D'(\mathbb{R})$ and $f \in C^\infty$. What is $(fT)'$?

A. not defined	B. fT'
C. $f'T$	D. $fT' + f'T$

4. What is the derivative of $T_{(x+1)\,\mathrm{sgn}\,x}$?

A. not defined	B. $x+\delta$
C. $2\delta_0+\mathrm{sgn}(x)$	D. something else

5. Is $\sum_{n=0}^{\infty}\delta_0^{(n)}$ a distribution on $D(\mathbb{R})$?

A. Yes	B. No

6. Is $\sum_{n=1}^{\infty}\delta_{\frac{1}{n}}$ a distribution on $D(\mathbb{R})$?

A. Yes	B. No

7. Is $\sum_{n=1}^{\infty}\delta_{\frac{1}{n}}$ a distribution on $D(\mathbb{R}\setminus\{0\})$?

A. Yes	B. No

8. Is $\sum_{n=1}^{\infty}(n!)^2\delta_n$ a distribution on $D(\mathbb{R})$?

A. Yes	B. No

9. Is there distributions of finite order but with $\sup_K C_K=\infty$?

A. Yes	B. No

10. Is there distributions with $\sup_K p_K = \infty$ but with $\sup_K C_K < \infty$?

A. Yes	B. No

11. Is there distributions with $\sup_K p_K = \infty$ and with $\sup_K C_K = \infty$?

A. Yes	B. No