This is a collection of examples that may be of use to students taking first year core maths modules for non-maths students at Warwick. These are examples that I made up for supervisions so may not be fully correct and may be way too easy or way too difficult - if you have any comments or corrections please send them to J.THOMAS.1@WARWICK.AC.UK. I have marked with (!) questions that are probably quite difficult for the particular module.

2019 Questions

Week 2: (worked in groups on the board 3-5 mins per question and then rotated)

1. Show that $ab \leq \frac{1}{2}(a^2 + b^2)$ for all $a, b \in \mathbb{R}$.
   Extension: For $\varepsilon > 0$, show that $ab \leq \frac{1}{2}(\varepsilon a^2 + \frac{1}{\varepsilon}b^2)$ for all $a, b \in \mathbb{R}$.
2. Sketch the graph of $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x+1| - 2|x| + |x-1|$.
   Extension: Sketch the graph of $g: \mathbb{R} \to \mathbb{R}$ given by $g(x) = ||x+1| - 2|x||$.
3. Show that $\sum_{k=1}^{n} k = \frac{n}{2}(n+1)$ for all $n \in \mathbb{N}$ with $n \geq 1$.
4. Show that $\sum_{k=1}^{n} k^2 = \frac{n}{6}(n+1)(2n+1)$ for all $n \in \mathbb{N}$.
5. Show that $10^n - 1$ is divisible by 9 for all $n \in \mathbb{N}$ with $n \geq 1$. Use this to answer question 36 from 2018.
6. Show that $n^3 + 3n^2 + 2n$ is divisible by 6 for all $n \in \mathbb{N}$ with $n \geq 1$.

Possible extra questions to look at: questions 2 and 37 from 2018.

Week 3: (worked individually for 2-3 mins, shared ideas and then completed the questions individually)

7. Suppose p is a prime that divides n^3 for some $n \in \mathbb{N}$. Does p divide n? Can you give a proof? Is this statement true if n is replaced with any natural number?
8. Suppose that $x, y \in \mathbb{Z}$ and that $x^3 - 6 = y^2$. Show that
   (i) y is odd
   (ii) y is not divisible by 3.
9. Show that $\sqrt{2} \notin \mathbb{Q}$.
   Extension: Show that $\sqrt{n} \in \mathbb{Q}$ if and only if n is a perfect square.
10. Suppose $n < N$ are real numbers. Find all $x \in \mathbb{R} \setminus \{-N, -n\}$ such that $\frac{1}{x+n} < \frac{1}{x+N}$.
11. For real numbers $a_1, \ldots, a_n \in \mathbb{R}$, show that $a_1 \cdots a_n > 0$ if and only if an even number of the a_j are negative.

Week 4: (went through assignment sheet then worked as a whole group on the board)

- Wrote down the following words on the board: increasing, decreasing, bounded below, bounded above, convergent, $a_n \to a$, null, tends to infinity
- took it in turns to write up a definition underneath the relevant word (the group either agreed or disagreed and we discussed the confusion and drew pictures to explain the meaning of the definitions)
- added the words: monotonic, bounded, strictly increasing, strictly decreasing and tends to minus infinity
- took it in turns to pick a definition and come up with an example of a sequence satisfying this definition (and where necessary proved or disproved this!)
- discussed which definitions implied other definitions and showed proofs where necessary:
  - convergent $\implies$ bounded

  - increasing $\implies$ bounded below
  - decreasing $\implies$ bounded above
  - bounded above and increasing $\implies$ convergent

- discussed which definitions do not necessarily imply other definitions:
  - tends to infinity $\not\Longrightarrow$ increasing
  - strictly increasing $\not\Longrightarrow$ tends to infinity

Week 5: (mainly went through assignment questions as a group)

12. Write $0.1\overline{23}$ as a fraction in its simplest form.
13. Question 5 from 2018.
14. Question 6 from 2018.

Week 6: (solved the following questions individually with the paper folding thing!)

15. Show that if $p \geq 5$ is a prime, then $p^2 - 1$ is divisible by 24. (This question is quite tricky!)
16. Suppose that $a_0 = 1$ and $a_{n+1} = 1 + \frac{1}{a_n}$. Assuming that the sequence (a_n) converges, find the limit a. Notice that this is not a proof that the sequence converges because you have assumed this in order to find the limit. Think about how you would actually prove that the sequence converges to this limit (this question may become easier in the next few weeks!)
17. Let $a_n = \left(\frac{2}{3}\right)^n$ and $b_n = \frac{n^n}{n!}$. Do these sequences converge? If so, find the limits.
18. For $p \neq q$ primes, calculate the following: (i) $\mathrm{lcm}(p^2, pq)$, (ii) $\mathrm{hcf}(2p, pq)$, and (iii) $\mathrm{lcm}(p^q, q^p)$.
19. Prove that none of the following statements can hold: (i) $\sqrt{-1} < 0$, (ii) $\sqrt{-1} = 0$, (iii) $\sqrt{-1} > 0$.

Week 8

For fixed $\alpha > 0$ consider the following sequence of real numbers

$$a_n = \frac{\alpha^n n!}{n^n} \quad \text{for } n \in \mathbb{N}.$$

You have seen that if $\alpha = 1$ then (a_n) is a null sequence. You did this by applying the ratio lemma and using the inequality

$$\left(1 + \frac{1}{n}\right)^n \geq 2 \tag{1}$$

(or you could have noted that $a_n \leq \frac{1}{n}$). We now try and generalise this result to the case $\alpha \neq 1$.

Q1: Use the fact that $(1 + x)^n \geq 1 + nx$ for all $x > -1$ to show that eq. (1) is satisfied for all $n \in \mathbb{N}$.

Q2: Use ratio test together with eq. (1) to show that (a_n) is null for all $\alpha < 2$.

Q3: Assuming that $(1 + \frac{1}{n})^n \to a$ as $n \to \infty$ (this is true! see Q5–Q7 below), show that (a_n) is null for all $\alpha < a$ and that (a_n) tends to $+\infty$ for all $\alpha > a$.

Q4: It turns out that $a = e \sim 2.71828....$ (which we will not prove here!). Use the following *Stirling's approximation* (which is a very involved result that we will not prove!) to determine the limit of (a_n) in the case that $\alpha = e$

$$\sqrt{2\pi}\sqrt{n}\left(\frac{n}{e}\right)^n \leq n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n.$$

Extra Questions

Q5: Show that $(1+\frac{1}{n})^n$ is increasing.

Q6: Show that $(1+\frac{1}{n})^n$ is bounded above.

Q7: Can you conclude that $(1+\frac{1}{n})^n$ converges as $n \to \infty$?

Week 10

20. Suppose that $\sum_{n=0}^{\infty} a_n$ is convergent. Show that $a_n \to 0$ as $n \to \infty$. Does $\sum_{n=1}^{\infty}(1+\frac{1}{n})^n$ converge?
21. Suppose that $\sum_{n=0}^{\infty} a_n$ is convergent. For $k \in \mathbb{N}$, does the summation $\sum_{n=k}^{\infty} a_n$ converge? It may be useful to first prove that
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{k-1} a_n + \sum_{n=k}^{\infty} a_n.$$
22. Suppose that $\sum_{n=0}^{\infty} a_n$ is convergent. Show that the sequence defined by $b_k := \sum_{n=k}^{\infty} a_n$ satisfies $b_k \to 0$ as $k \to \infty$. This is sometimes stated as: the tail of a convergent summation tends to zero.

Possible extra questions to look at: Q12, Q17, Q18 from 2018

2018 Questions

1. Show that, for all $n \in \mathbb{N}$, the number $n^3 + 3n^2 + 2n$ is divisible by 6.
2. Show that $2^{2n} - 1$ is divisible by 3 and that $2^{3n} - 1$ is divisible by 7 for all $n \in \mathbb{N}$. Can your proof be generalised to a statement about $2^{kn} - 1$ for fixed $k \in \mathbb{N}$? Give two factors of $2^{2018} - 1$.
3. Suppose $P(n)$ is a statement about natural numbers $n \in \mathbb{N}$ with the property that if $P(n)$ is true then $P(n^2+1)$ is also true. Further suppose that $P(1)$ is true and $P(101)$ is false. Match up the statements with the descriptions

$P(3)$	False
$P(5)$	Haven't got enough information
$P(101^2+1)$	True

Explain your answers with full justifications.

4. Use the fact (can you prove it?) that $a \equiv b \pmod{n}$ implies $ca \equiv cb \pmod{n}$ for any $c \in \mathbb{N}$, to show that $(n+1)^{n+1} \equiv 1 \pmod{n}$ for all $n \in \mathbb{N}$. Another way of phrasing the above question is: Does n divide $(n+1)^{n+1} - 1$ for all $n \in \mathbb{N}$? Try to prove this by writing $(n+1)^{n+1}$ out as a binomial expansion. Another possible proof involves using the geometric series formula:
$$\frac{(n+1)^{n+1}-1}{n} = \frac{(n+1)^{n+1}-1}{(n+1)-1} = \sum_{k=0}^{n}(n+1)^k.$$
Use this formula to show that n divides $(n+1)^{n+1} - 1$. Moreover show that $\frac{(n+1)^{n+1}-1}{n}$ is odd for all $n \in \mathbb{N}$. Use this to explain why $(n+1)^{n+1} \not\equiv 1 \pmod{2n}$. What is $(n+1)^{n+1}$ modulo $2n$? What is $\frac{(n+1)^{n+1}-1}{n}$ modulo n? How much of this whole question changes if we replace $(n+1)^{n+1}$ with $(n+1)^r$ for some $r \in \mathbb{N}$? Is
$$\left(\left(2^2\right)^2\right)^2 - 1$$
prime?

5. Fix $k \in \mathbb{N}$, $k > 0$. True or false: There exists unique natural numbers n, m such that

$$n < m, \quad \mathrm{hcf}(n,m) = 1, \quad \text{and} \quad \mathrm{lcm}(n,m) = 6^k.$$

How many pairs (n, m) are there such that

$$\begin{aligned} &(a) \quad n < m, \quad \mathrm{hcf}(n,m) = 6, \quad \text{and} \quad \mathrm{lcm}(n,m) = 6^k?\\ &(b) \quad n < m, \quad \mathrm{hcf}(n,m) = 5, \quad \text{and} \quad \mathrm{lcm}(n,m) = 6^k?\\ &(c) \quad n < m, \quad \mathrm{hcf}(n,m) = 9, \quad \text{and} \quad \mathrm{lcm}(n,m) = 12345? \end{aligned}$$

6. How many of the following complex numbers are real and how many are purely imaginary?

$$0, \quad \mathrm{Im}(1-i), \quad \text{solutions to: } z = -\frac{1}{z}, \quad \sqrt{-1}^{\sqrt{-1}}, \quad i^{z+1} + i^{z-1}, \quad i^{56}, \quad \text{and} \quad \sum_{k=0}^{123} (-i)^k.$$

7. Show that none of the following statements hold

$$i < 0, \qquad i = 0, \qquad i > 0.$$

8. Suppose $p \neq q$ are primes. Write down the following quantities:

$$\mathrm{lcm}(p^2, pq), \qquad \mathrm{hcf}(2p, 3pq), \qquad \mathrm{lcm}(p^p, q^q).$$

9. Find the limits of the following sequences:

$$\frac{n^n}{n!}, \qquad \frac{2^n}{3^n}, \qquad \frac{2^n n^{10} n!}{n^n}.$$

10. Adapt the proof that $\sqrt{2}$ is irrational (from lectures) to prove: For $n \in \mathbb{N}$, the square root $\sqrt{n}$ is irrational if and only if n is not a perfect square. This is of course equivalent to: $\sqrt{n} \in \mathbb{Q}$ if and only if $n = m^2$ for some $m \in \mathbb{N}$.

11. Give examples of isosceles right-angled triangles with exactly 1, 2 and 3 irrational side lengths. Show that every isosceles right-angled triangle has at least one irrational side length.

12. In this question, we will show that the sequence defined by

$$b_1 = 2, \quad b_{n+1} = 1 + \frac{1}{b_n} \quad \text{for } n > 1$$

is convergent and we'll also find the limit. We say a sequence $(a_n)_{n \in \mathbb{N}}$ is a contraction if there exists a constant $M \in (0, 1)$ such that

$$|a_{n+1} - a_n| \leq M |a_n - a_{n-1}| \quad \text{for all } n > 1.$$

By showing that $b_n \geq \frac{3}{2}$, prove that the sequence (b_n) is a contraction (with constant $M = \left(\frac{3}{2}\right)^{-2}$) and hence conclude that it is a Cauchy sequence. Does (b_n) necessarily converge? What is the limit? If we instead set $b_1 = 1$, does the sequence still converge? What is the limit? Notice that this question gives meaning to the *continued fraction*

$$1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}.$$

13. (!) Let us fix a bounded sequence of real numbers (a_n) and $a \in \mathbb{R}$. Further suppose that the sequence (a_n) satisfies the following property:

$$\textbf{(P)} \quad \Big((a_{n_k}) \text{ is a convergent sub-sequence and } (a_{n_k}) \to b \text{ for some } b \in \mathbb{R} \Big) \implies (b = a)$$

The above statement says that **any** convergent sub-sequence converges to the **same** value. We will now show that this is enough to conclude that the whole sequence converges and that $a_n \to a$ as $n \to \infty$.

(i) Suppose for the sake of contradiction that $a_n \not\to a$ as $n \to \infty$. Write down the definition of $a_n \not\to a$ as $n \to \infty$.

(ii) Show that there exists a $\varepsilon > 0$ and a sub-sequence (a_{n_k}) such that $|a_{n_k} - a| > \varepsilon$ for all $k \in \mathbb{N}$.

(iii) Setting $b_k := a_{n_k}$ for $k \in \mathbb{N}$, defines a new sequence of real numbers. Explain why (b_k) is a bounded sequence.

(iv) Use the Bolzano-Weierstrass Theorem to show there exists a sub-sequence (b_{k_j}) and a $b \in \mathbb{R}$ such that $b_{k_j} \to b$ as $j \to \infty$.

(v) Using **(P)**, explain why $b = a$.

(vi) Show that this contradicts your initial assumption and conclude that the whole sequence (a_n) converges to the limit a.

14. For fixed $x \in \mathbb{R}$, consider the real valued sequence defined by

$$a_1 = x, \quad a_{n+1} = \sqrt{1 + a_n} \quad \text{for } n > 1$$

For $x = 1$, show that this sequence is bounded and increasing. Conclude that (a_n) converges to some $a \in \mathbb{R}$. What is a? Suppose instead that $x = 10$. Does the sequence still converge? If so, what is the limit? What other *initial conditions* x can you choose such that sequence converges and has limit a? Notice that this question gives meaning to the quantity

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}}}}.$$

15. (!) For fixed $x \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{N}$, consider the sequence defined by

$$a_1 = x, \qquad a_{n+1} = \alpha \left(a_n\right)^{\beta}.$$

If $\beta = 1$, it can be shown that $a_{n+1} = \alpha^n x$. Can you generalise this formula to the case where $\beta > 1$? Prove your formula by induction. *If you get stuck, you could try the following approach: make the assumption that $a_n = \alpha^{f(n)} x^{g(n)}$ for some functions f and g. Now you can use the recursion formula to solve for f and g. To start, we know that $f(1) = 0, g(1) = 1$.*

16. (Cesàro summation) We say that a sequence (a_n) is *Cesàro summable* if the sequence $\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)_{n\in\mathbb{N}}$ converges. Give an example of a sequence that is Cesàro summable but doesn't converge to any value $a \in \mathbb{R}$. Give an example of a sequence that is not Cesàro summable. Show that if (a_n) is a convergent sequence with limit a, then it is Cesàro summable. Moreover, show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_n = a.$$

You have shown that Cesàro summability is weaker than our *usual* notion of convergence for sequences of real numbers.

17. In your assignments, you have shown that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1.$$

The purpose of this question is to generalise this result. For $n, m \in \mathbb{N}$, show that

$$\frac{1}{n(n+m)} = \frac{1}{m}\left(\frac{1}{n} - \frac{1}{n+m}\right).$$

Use this to calculate

$$S_{N,m} := \sum_{n=1}^{N} \frac{1}{n(n+m)}$$

for fixed $m, N \in \mathbb{N}$. Show that

$$\sum_{n=1}^{\infty} \frac{1}{n(n+m)} = \frac{1}{m} \sum_{k=1}^{m} \frac{1}{k}.$$

18. Use Question 17 to evaluate (i) & write (ii) as a finite sum:

$$\text{(i)} \quad \sum_{n=2}^{\infty} \frac{1}{n^2-1} \qquad \text{(ii)} \quad \sum_{n=m+1}^{\infty} \frac{1}{n^2-m^2} \qquad \text{for } m \in \mathbb{N}.$$

Use (i) to show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{2}$$

is convergent. Further, using the fact (which we will not prove!) that the value of (2) is $\frac{\pi^2}{6}$, show that $2\pi \leqslant \sqrt{42}$. Prove (using any method you like) that $4\sqrt{2} \leqslant 2\pi \leqslant \sqrt{42}$.

19. Given a real valued sequence $(b_n)_{n\in\mathbb{N}}$ define another sequence by

$$a_1 = 2018 \qquad a_{n+1} = \frac{n}{n+1} a_n + \frac{1}{n+1} b_{n+1}, \text{ for } n \in \mathbb{N}.$$

Write a_n as a function of the sequence $(b_n)_{n\in\mathbb{N}}$ and thus conclude that if (b_n) is Cesàro summable (see Question 16) then (a_n) converges.

20. For a given field K the set of polynomials over K can be defined to be

$$K[X] := \{a_0 + a_1 X + \cdots + a_N X^N : N \in \mathbb{N} \cup \{0\}, \quad a_j \in K \quad \text{for } 0 \leqslant j \leqslant N\}$$

where X is an abstract formal variable satisfying $X^0 = 1$ and $X^a X^b = X^{a+b}$ for all $a, b \in \mathbb{N} \cup \{0\}$. We can define addition and multiplication in the natural way (can you write this down rigorously?) and so the set $K[X]$ can be thought of as the smallest set that contains all the elements of K and the formal variable X such that

$$P, Q \in K[X] \implies P + Q, PQ \in K[X].$$

Given a polynomial $a_0 + a_1 X + \cdots + a_N X^N \in K[X]$, we can define a corresponding polynomial mapping $f\colon K \to K$ by $x \mapsto a_0 + a_1 x + \cdots + a_N x^N$. We can label with Φ, the function that takes a polynomial and gives the corresponding polynomial mapping:

$$\Phi : K[X] \to \{f\colon K \to K \text{ such that } f(x) = a_0 + a_1 x + \cdots + a_N x^N \text{ for some } N \in \mathbb{N} \cup \{0\} \text{ and } a_j \in K\}$$
$$\Phi(a_0 + \cdots + a_N X^N) = f \quad \text{where } f : K \to K \text{ and } f(x) = a_0 + \cdots + a_N x^N.$$

Show that, if $K = \mathbb{R}$, then Φ as defined above is a bijection. Find an example of a field K and two distinct polynomials that give rise to the same polynomial mapping. In other words, find two polynomials $a_0 + \cdots + a_N X^N \neq b_0 + \cdots + b_M X^M \in K[X]$ such that

$$a_0 + \cdots + a_N x^N = b_0 + \cdots + b_M x^M, \quad \text{for all } x \in K$$

(for some suitable field K). In other words, show that, for some choice of field K, the mapping Φ (as defined above) is not injective. *Hint: consider the field of two elements and use the fact that $x^2 = x$ in this field. What other fields can you consider?*

Extra Questions: (20 Dec 2018)

21. If you have done question 1, try this: Show that $n^3 + 30n^2 + 299n + 990$ is divisible by 6 and that $8n^3+36n^2+52n+24$ is divisible by 12. What is the largest $k \in \mathbb{N}$ you can find such that $n^4+6n^3+11n^2+6n$ is divisible by k for all $n \in \mathbb{N}$?

22. Following on from question 2, show that the your proof can be used to show the following statement: Let $x \in \mathbb{N}$. Then if k divides n, then $x^k - 1$ divides $x^n - 1$.

23. Looking at question 4, can you adapt the ideas in your proof to prove the *hockey-stick identity*:

$$\binom{n+1}{k+1} = \sum_{j=k}^{n} \binom{j}{k}.$$

Why is this called the hockey-stick identity? *HINT: You may wish to consider $\frac{(x+1)^{n+1}-1}{x}$ in two different ways.*

24. Show that for $n, m \in \mathbb{N}$, we have $\mathrm{hcf}(n, m)$ divides $\mathrm{lcm}(n, m)$.

25. Suppose that
$$n = p_1^{i_1} \cdots p_r^{i_r} \quad \text{and} \quad m = p_1^{j_1} \cdots p_r^{j_r}$$
for distinct primes $p_1, \cdots, p_r$ and integers $i_1, j_1, \cdots i_r, j_r \geq 0$ (for some $r \in \mathbb{N}$). Show that
$$\mathrm{hcf}(n, m) = p_1^{\min\{i_1, j_1\}} \cdots p_r^{\min\{i_r, j_r\}} \quad \text{and} \quad \mathrm{lcm}(n, m) = p_1^{\max\{i_1, j_1\}} \cdots p_r^{\max\{i_r, j_r\}}.$$

26. The geometric series formula for finite summations with the complex ratio $z \in \mathbb{C} \setminus \{0\}$ is:
$$\sum_{k=0}^{n} z^k = \frac{z^{n+1} - 1}{z - 1}.$$
You may have only used/proved this statement for $z \in \mathbb{R} \setminus \{0\}$ but does the proof change when we replace z with a non-zero complex number?

27. Let us consider $k > 0$, and the sequence $d_n = \frac{k^n n!}{n^n}$ for $n \in \mathbb{N}$. Show that (d_n) converges for $k < e$ and diverges for $k > e$. Use the following *Stirling's approximation* to determine whether (d_n) converges or diverges when $k = e$:
$$\sqrt{2\pi}\sqrt{n}\left(\frac{n}{e}\right)^n \leq n! \leq e\sqrt{n}\left(\frac{n}{e}\right)^n.$$

28. Prove the following fact: Let $n = p_1^{k_1} \cdots p_m^{k_m}$ where p_j are distinct primes and k_j are positive integers. Then n is a perfect square if and only if all the k_j are even.

29. A finite continued fraction is of the form
$$[a_0; a_1, a_2, \cdots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}$$
for some $n \in \mathbb{N}$, $a_0 \in \mathbb{Z}$ and $a_j \in \mathbb{N}$ for all $1 \leq j \leq n$.
Show that $1.234 = [1; 4, 3, 1, 1, 1, 10]$ and $4.321 = [4; 3, 8, 1, 2, 12]$.
Show that $[a_0; a_1, a_2, \cdots, a_n, 1] = [a_0; a_1, a_2, \cdots, a_n + 1]$.
Show that the set of finite continued fractions is exactly the set of rational numbers.

30. Does the following sequence converge? If so, what is the limit?
$$b_1 = 2, \quad b_{n+1} = \sqrt{2 + b_n} \quad \text{for } n > 1.$$
What about if b_1 was an arbitrary real number?

31. Show that there exists irrational numbers $a, b \notin \mathbb{Q}$ such that a^b is rational.

32. Show that the set of sequences of natural numbers is uncountable. That is, show that the set
$$A := \{(a_n)_{n \in \mathbb{N}} \colon a_j \in \mathbb{N} \quad \forall n \in \mathbb{N}\}$$
is uncountable. Is the set
$$B := \{(b_n)_{n \in \mathbb{N}} \colon b_j \in \{0, 1\} \quad \forall n \in \mathbb{N}\}$$
countable?

33. Show that the set of eventually zero sequences of natural numbers is countable. That is, the set
$$A := \{(a_n)_{n \in \mathbb{N}} \colon a_j \in \mathbb{N} \quad \forall j \in \mathbb{N} \quad \text{and} \quad \exists N \text{ s.t. } a_n = 0 \quad \forall n > N\}$$
is countable.
HINT: it may be easier to show that $\bigcup_{n=0}^{\infty} A_n$ is countable if each A_n is countable. Why is this sufficient?

34. Show that there is no surjective mapping from a set to its power set. That is, the power set has strictly larger cardinality.

35. (Expansions in Base n) Let $x \in [0,1]$ and $n \in \mathbb{N}$, $n \geq 2$. If there exists a sequence $(x_k)_{k\in\mathbb{N}}$ such that $x_k \in \{0, \cdots, n-1\}$ and
$$x = \sum_{k=1}^{\infty} x_k n^{-k}$$
we say that this summation is the base n expansion of x. For example, with $n = 10$, this is the normal decimal (base 10) expansion and, for $n = 2$, this is the binary (base 2) expansion. Notice that this expansion is not unique: For example, in base 10, we have
$$0.4 = 0.3999999999999999999\cdots$$
   (a) Given any sequence $(x_k)_{k\in\mathbb{N}}$ with $x_k \in \{0, \cdots, n-1\}$ for all $k \in \mathbb{N}$, show that $\sum_{k=1}^{\infty} x_k n^{-k}$ converges to some $x \in [0,1]$.
   (b) For what $x \in [0,1]$ is the decimal expansion of x not unique?
   (c) For what $x \in [0,1]$ is the base n expansion of x not unique?
   (d) Does every $x \in [0,1]$ have a base n expansion?

36. Show that $n \in \mathbb{N}$ is divisible by 9 if and only if the sum of the digits of n (in the decimal expansion) is divisible by 9.

37. Show that $\sum_{k=0}^{N} x_k 10^k$ is divisible by 11 if and only if $\sum_{k=0}^{N} x_k (-1)^k$ is divisible by 11. Hence show that every palindromic number with an even number of digits is divisible by 11. Notice that 11 is the only palindromic prime with an even number of digits.

Extra Question: (15 Mar 2019)

38. Show that if $p \geq 5$ is a prime then $p^2 - 1$ is divisible by 24. Can your proof be used to extend the statement to all integers $n \geq 5$ such that 6 does not divide n?