This is a collection of examples that may be of use to students taking first year core maths modules for non-maths students at Warwick. These are examples that I made up for supervisions so may not be fully correct and may be way too easy or way too difficult - if you have any comments or corrections please send them to J.THOMAS.1@WARWICK.AC.UK. I have marked with (!) questions that are probably quite difficult for the particular module.

Linear Algebra

Example 1: Change of Basis

Let us consider the linear map $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$T((x,y,z)) = (2x, y-z, z)$$

Let $E = \{e_1, e_2, e_3\}$ denote the standard basis of $\mathbb{R}^3$.

(a) Write down the matrix of T with respect to E. We shall denote this by $[T]_E^E$.

Let

$$F = \left\{ f_1 = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, f_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, f_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}. \right\}$$

(b) Check that F is a basis of $\mathbb{R}^3$.

(c) Write the matrix that represents T with respect to the basis F.

Another way of looking at this is the following: If we wish to express T with respect to F, we are interested in the mapping $T : (\mathbb{R}^3, F) \to (\mathbb{R}^3, F)$. We can see that this is the composition of the following linear maps:

$$\begin{aligned} \mathrm{id} &: (\mathbb{R}^3, F) \to (\mathbb{R}^3, E) \text{ with associated matrix } [\mathrm{id}]_E^F, \\ T &: (\mathbb{R}^3, E) \to (\mathbb{R}^3, E) \text{ with associated matrix } [T]_E^E \text{ and} \\ \mathrm{id} &: (\mathbb{R}^3, E) \to (\mathbb{R}^3, F) \text{ with associated matrix } [\mathrm{id}]_F^E. \end{aligned}$$

Therefore, we can see that the map $T : (\mathbb{R}^3, F) \to (\mathbb{R}^3, F)$ has the following associated matrix

$$[T]_F^F = [\mathrm{id}]_F^E \, [T]_E^E \, [\mathrm{id}]_E^F$$

This is simply because the composition of linear maps corresponds to matrix multiplication of the associated matrices. Note that the matrices $[\mathrm{id}]_F^E$ and $[\mathrm{id}]_E^F$ are change of basis matrices.

(d) Show that

$$[\mathrm{id}]_F^E = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \\ -1/2 & 1 & -3/2 \end{pmatrix} \text{ and } [\mathrm{id}]_E^F = \begin{pmatrix} 2 & -1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

(e) Hence show that

$$[\mathrm{id}]_F^E \, [T]_E^E \, [\mathrm{id}]_E^F = \begin{pmatrix} 2 & -1/2 & 0 \\ 0 & 1 & 0 \\ -1 & -1/2 & 1 \end{pmatrix}$$

(f) Compare (c) and (e)

(g) Use your change of basis matrices to write $v = e_1 + 2e_2 + 3e_3$ in terms of the basis F and $w = f_1 + 2f_2 + 3f_3$ in terms of the basis E.

(h) Check (g) by solving the equations $v = \alpha f_1 + \beta f_2 + \gamma f_3$ and $w = ae_1 + be_2 + ce_3$ for the coefficients $\alpha, \beta, \gamma, a, b, c$.

(i) Use what we have done to calculate the matrix representation of T with respect to E in the domain and F in the range. That is, find the matrix representing the linear map $T : (\mathbb{R}^3, E) \to (\mathbb{R}^3, F)$. Do the same for the map $T : (\mathbb{R}^3, F) \to (\mathbb{R}^3, E)$.

Question 2: Solving a Linear System of Equations

Let us consider the finite sequence $a_1 = 6, a_2 = 17, a_3 = 34$. You may be familiar with finding the n^{th} term of this sequence but you may not be familiar with the following method:

(a) What are the possible degrees n such that there exists a polynomial, p, of degree n such that $p(n) = a_n$ for $n = 1, 2, 3$?

(b) Set up a linear system of equations for the coefficients of a polynomial p of least degree as in (a).

(c) Solve the linear system and thus find p.

(d) Check your solution by evaluating $p(n)$ for $n = 1, 2, 3$.

Question 3: Matrix Diagonalisation - An Application

Let us consider the *Fibonacci* sequence defined by $F_0 = 0, F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 1$. You may know that the n^{th} term in this sequence is given in terms of the golden ratio - follow the below steps to see a proof of this fact!

(a) Prove by induction that for $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, we have

$$A^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}.$$

(b) Find the eigenvalues λ_1, λ_2 and corresponding eigenvectors v_1, v_2 of A.

(c) Hence, find an invertible matirx S such that $A = S \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} S^{-1}$.

(d) Notice that $A^n = S \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}^n S^{-1}$ and hence show that

$$F_n = \frac{1}{\sqrt{5}} \left(\phi^n - \psi^n \right)$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio and $\psi = \frac{1-\sqrt{5}}{2}$.

(extra) Try to use the matrix A to show that if n divides m then F_n divides F_m.