Linear Algebra: Vector Spaces

True or False?
$v_1, \dots, v_n \in V$ are linearly dependent if and only if there exists $j \in \{1, \dots, n\}$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ such that

$$v_j = \sum_{k\colon k \neq j} \alpha_k v_k.$$

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True or False?
The vectors $(1, 1, 1)$, $(-1, 0, 1)$, $(0, 1, 0) \in \mathbb{R}^3$ form a basis of $\mathbb{R}^3$ (here, $\mathbb{R}^3$ is a vector space over $\mathbb{R}$)

Linear Algebra: Vector Spaces

True or False?
The vectors
$v_1 = (1, 2, 3), v_2 = (1, 3, 2), v_3 = (2, 1, 3)$
$v_4 = (2, 3, 1), v_5 = (3, 1, 2), v_6 = (3, 2, 1)$ are linearly independent in $\mathbb{R}^3$.

Linear Algebra: Vector Spaces

True or False?
For $n \in \mathbb{N}$, any set of $n + 1$ vectors in $\mathbb{R}^n$ is not linearly independent.

Linear Algebra: Vector Spaces

True or False?
For $n \in \mathbb{N}$, any set of $n + 1$ vectors in $\mathbb{R}^n$ spans.

Linear Algebra: Vector Spaces

True or False?
V is a vector space. If the set $v_1, \dots, v_n \in V$ is linearly independent but does not span V, then there exist a $v \in V$ such that $v_1, \dots, v_n, v$ are linearly independent.

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True or False?
$1, 1 - x, 1 - x^2, \dots, 1 - x^n \in \mathbb{R}[x]_{\leq n}$ form a basis.

Linear Algebra: Vector Spaces

True or False?
$\mathbb{C}^n$ is a vector space over $\mathbb{R}$ of dimension $2n$ (there exists a basis of $(\mathbb{C}^n, \mathbb{R})$ of $2n$ vectors).

Linear Algebra: Vector Spaces

True or False?
$\{(1 + x)^k \colon k = 0, \dots, n\} \subset \mathbb{R}[x]_{\leq n}$ forms a basis of $\mathbb{R}[x]_{\leq n}$.

Linear Algebra: Vector Spaces

True or False?
For any finite set A of vectors spanning a vector space V, there exists a subset $\{v_1, \dots, v_n\} \subset A$ which is a basis of V.

True.
For an arbitrary $(x, y, z) \in \mathbb{R}^3$, we may solve $(x, y, z) = \alpha(1, 1, 1) + \beta(-1, 0, 1) + \gamma(0, 1, 0)$ which gives $\alpha = \frac{1}{2}(x+z)$, $\beta = \frac{1}{2}(x-z)$ and $\gamma = y - \frac{1}{2}(x+z)$. This shows that the vectors span the space.
Now, we need to explain why the vectors are linearly independent. Either we can write $\alpha(1, 1, 1) + \beta(-1, 0, 1) + \gamma(0, 1, 0) = 0$ and show that $\alpha = \beta = \gamma = 0$ or we can say that any set of n vectors that span an n dimensional vector space must also be linearly independent (and hence a basis) .

True.
The vectors are not linearly independent (i.e. are linearly dependent) if and only if there exists $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ which are not all zero such that

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$

Letting j be such that $\alpha_j \neq 0$, we obtain

$$v_j = -\frac{1}{\alpha_j} \sum_{k:\ k \neq j} \alpha_k v_k$$

.

True.
Seen in lectures a more general result. .

False.
$\mathbb{R}^3$ has dimension 3 and so any set of more than 3 vectors cannot be linearly independent. .

True.
Since the set does not span V, there exists a $v \in V$ that cannot be written as a linear combination of $v_1, \dots, v_n$. Now, if $\alpha_1 v_v + \dots \alpha_n v_n + \alpha v = 0$ then $\alpha = 0$ (if not $v = -\frac{1}{\alpha}(\alpha_1 v_v + \dots \alpha_n v_n)$ contradicting the fact v is not in the span of the v_i) and so $\alpha_1 v_v + \dots \alpha_n v_n = 0$ which implies all $\alpha_i = 0$ by linear independence of the v_i. This means $v_1, \dots, v_n, v$ is linearly independent .

False.
e.g. take $v \in V$, the set $v, 2v, 3v, \dots, (n+1)v$ has the same span as v (which is the whole space if and only if $n = 1$) .

True! We let $e_1, \dots, e_n$ is the standard basis of $\mathbb{R}^n$. We claim that $E = \{e_1, \dots, e_n, ie_1, \dots, ie_n\}$ is a basis of $(\mathbb{C}^n, \mathbb{R})$. For $z \in \mathbb{C}^n$, we can find real vectors $x, y \in \mathbb{R}^n$ such that $z = x + iy$ and since $e_1, \dots, e_n$ is a basis of $\mathbb{R}^n$, we can conclude that E spans $(\mathbb{C}^n, \mathbb{R})$. Similarly, supposing that $\alpha_1 e_1 + \cdots + \alpha_n e_n + \beta_1 ie_1 + \cdots + \alpha_n ie_n = 0$ for $\alpha_j, \beta_j \in \mathbb{R}$. $0 \in \mathbb{C}$ is the point where both real and imaginary components are zero. Therefore, we have $\alpha_1 e_1 + \cdots + \alpha_n e_n = 0$ and $\beta_1 e_1 + \cdots + \beta_n e_n = 0$ thus, by the linear independence of the standard vectors, $\alpha_j = \beta_j = 0$ for all j. That is, E is lin. independent over $\mathbb{R}$.

True.
A general polynomial can be written as $p(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0$ which is also $p(x) = -\alpha_n(1-x^n) - \cdots - \alpha_1(1-x) + \alpha_0 + \alpha_1 + \cdots + \alpha_n$ and so the set of polynomials span the space. They are linearly independent because there is n spanning vectors in an n dimensional space .

True.
We shift the set A to get a set of linearly independent vectors which also span by the assumption that A spans .

True.
Let $v_k = (1+x)^k$ for $k = 0, \dots, n$. $\mathbb{R}[x]_{\leq n}$ is of dimension $n+1$ so we only need to show that $v_0, \dots, v_n$ spans. We show this by induction on n. For $n = 0$, $\mathbb{R}[x]_{\leq 0} = \mathbb{R}$ and 1 spans the space. Suppose that $v_0, \dots, v_{n-1}$ spans $\mathbb{R}[x]_{\leq n-1}$. Now for $p(x) = \alpha_n x^n + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{R}[x]_{\leq n}$, we can write $p(x) = \alpha_n(1+x)^n + q(x)$ where $q(x)$ is a polynomial of degree at most $n-1$ and so may be written as a linear combination of $v_0, \dots, v_{n-1}$ by induction .

Linear Algebra: Vector Spaces

True or False?
Let V be a finite dimensional vector space. For any set A of linearly independent vectors, there exists a superset $\{v_1, \ldots, v_n\} \supset A$ which is a basis of V.

Linear Algebra: Vector Spaces

True or False?
Let V be a finite dimensional vector space over $\mathbb{R}$. There exists infinitely many sets of linearly independent vectors.

Linear Algebra: Vector Spaces

True or False?
If $v_1, \ldots, v_n \in V$ forms a basis, then for any $\alpha_1, \ldots, \alpha_n \in \mathbb{K}$ the set of vectors $\alpha_1 v_1, \ldots, \alpha_n v_n$ forms a basis.

True.
If $f_1, \dots, f_n$ is a basis of V, then $\alpha f_1, \dots, \alpha f_n$ is a new basis of V for all $\alpha \in \mathbb{R} \setminus \{0\}$. .

True.
We let E be a basis of V and define $B = E \cup A$. Shifting B gives a super-set of A which is a basis of V. .

False.
One or more of the α_j may be zero which makes $\alpha_j v_j = 0 \in V$. .